



The Open
University

M303/J

M3031506F1



Module Examination 2015

Further pure mathematics

Monday 8 June 2015

10.00 am – 1.00 pm

Time allowed: 3 hours

The exam is in two parts. Attempt *all* questions in Part 1 (each question is worth 11 marks) and *two* out of the three questions in Part 2 (each question is worth 17 marks). If you submit attempts to all three questions in Part 2, your best two scores will count towards your result.

It should take about two hours to do the Part 1 questions and an hour to do two Part 2 questions.

Start each question on a new page.

When sketching sets, take care to indicate which points are included: use dashed lines to indicate boundaries that are not included and solid lines to indicate boundaries that are. Use small open circles $\circ$ to indicate any ‘corner’ points on a boundary that are not included and small closed circles $\bullet$ to indicate any ‘corner’ points that are included.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.

The use of calculators is <i>not</i> permitted in this examination.

PART 1

Attempt all questions. Each question is worth 11 marks. All of the Part 1 questions count towards your result.

Question 1 (11 marks, Book A)

- (a) Use the Principle of Mathematical Induction to prove that for all integers $n \geq 2$ the number $10^n - 4$ is divisible by 12. [3]
- (b) From the definition of congruence alone, prove the following, for integers a, b, c, d, k, n and r with $n > 1$ and $r > 0$.
- (i) If $a \equiv b \pmod{n}$ then $ka + c \equiv kb + c \pmod{n}$.
- (ii) If $ka \equiv b \pmod{n}$ and $kc \equiv d \pmod{n}$ then $ad \equiv bc \pmod{n}$.
- (iii) $ra \equiv rb \pmod{rn}$ if, and only if, $a \equiv b \pmod{n}$. [6]
- (c) Suppose that $n \equiv 3 \pmod{8}$. Determine the least positive residue of $5n + 3$
- (i) modulo 4;
- (ii) modulo 20. [2]

Question 2 (11 marks, Book B)

Consider the following six groups.

$$\mathbb{Z}_{12}, \quad \mathbb{Z}_9, \quad \mathbb{Z}_6 \times \mathbb{Z}_2, \quad \mathbb{Z}_4 \times \mathbb{Z}_3, \quad \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

- (a) (i) List the groups above that are cyclic.
- (ii) List the groups above that do not have an element of order 6.
- (iii) Write down any pairs of isomorphic groups from the list. [5]
- (b) Dic_4 is given by the presentation
- $$\text{Dic}_4 = \langle a, b \mid a^4 = e, a^2 = b^4, aba^3b = e \rangle,$$
- with elements expressed in the standard form as
- $$a^r b^s \text{ for } r = 0, \dots, 3, s = 0, \dots, 3.$$
- Express the element bab in standard form. [3]
- (c) Prove that no group of order 54 is simple. [3]

Question 3 (11 marks, Book C)

- (a) This part of the question is concerned with the σ -function (that is equal to the ‘sum of the positive divisors’). You may assume that this function is multiplicative. Note that if p is prime, then $\sigma(p) = p + 1$.

Suppose that a positive integer n has the property that $2n + \sigma(n)$ is divisible by 3.

- (i) Show that n cannot be prime. [1]
 - (ii) Show that if $n = p^2$ for some prime p , then $p \equiv 2 \pmod{3}$. [2]
 - (iii) Show that if $n = p^3$ for some prime p , then $p \equiv 1 \pmod{6}$. [3]
- (b) By considering the field $\mathbb{Z}_7$, prove that $8x^3 - 6x - 1$ is irreducible over $\mathbb{Z}$. [5]

Question 4 (11 marks, Books D and F)

Let

$$A = \{(a, b) \in \mathbb{R}^2 : b > 0\}, B = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 4\}$$

and

$$C = A \cap B.$$

- (a) Sketch on separate diagrams each of A , B and C taking care to indicate which points lie in the sets. [3]
- (b) Write down the closure, interior and boundary of C for the following metrics:
 - (i) the Euclidean metric for the plane
 - (ii) the product metric e on the plane given by
$$e((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + d_0(x_2, y_2),$$
where d_0 is the discrete metric for $\mathbb{R}$. [6]
- (c) Show that C is not compact for e . [2]

Question 5 (11 marks, Book E)

This question concerns the ring of Gaussian integers $\mathbb{Z}[i]$, with $i^2 = -1$. This ring is a principal ideal domain and a Euclidean domain. Consider the ideal J of $\mathbb{Z}[i]$ given by $J = \langle 5, 3 - i \rangle$.

- (a) By using the Euclidean Algorithm (or otherwise) determine all $z \in \mathbb{Z}[i]$ such that $J = \langle z \rangle$. [5]
- (b) Prove that $\mathbb{Z}[i]/J$ is a field. [6]

Question 6 (11 marks, Book F)

Let

$$A = \{f \in C[0, 1] : |f(x)| \leq x^2 \text{ for each } x \in [0, 1] \text{ and} \\ |f(x) - f(y)| \leq 2\sqrt{|x - y|} \text{ for } x, y \in [0, 1]\}$$

- (a) Use the Arzelà-Ascoli theorem to show that A is $d_{\max}$ -sequentially compact. [8]
- (b) Define $F: C[0, 1] \rightarrow \mathbb{R}$ by $F(f) = \int_0^1 f(t) \sin t \, dt$. Show that F attains a minimum on A . [3]

PART 2

Attempt two questions. Each question is worth 17 marks.
(If you submit attempts to all three questions, your best two scores will count towards your result.)

Question 7 (17 marks, topic: groups)

Let G be a group of order 44.

- (a) Find the possible numbers of Sylow 2-subgroups of G . [2]
- (b) Show that G has a normal subgroup of order 11. [3]
- (c) Show that if G has just one Sylow 2-subgroup, then G is abelian.
Write down the non-isomorphic possibilities for G in this case. [5]
- (d) Consider the dihedral group D_{22} of order 44 (the symmetry group of a regular 22-gon).
 - (i) Find the number of Sylow 2-subgroups of D_{22} .
 - (ii) Give a brief description of the Sylow 2-subgroups of D_{22} in terms of the symmetries of a 22-gon.
 - (iii) Find the elements of the Sylow 2-subgroups of D_{22} in standard form $(r^i s^j$ for $i = 0, \dots, 10, j = 0, 1)$. [7]

Question 8 (17 marks, topic: number theory and rings)

- (a) Prove that there are infinitely many primes of the form $3k + 2$, where k is a non-negative integer, by the following method.
Suppose that there are only a finite number of such primes, say $p_1, p_2, \dots, p_r$. Reach a contradiction by showing that the number $(p_1 p_2 \dots p_r)^2 + 1$ must have a prime divisor of the form $3k + 2$ and that this cannot be one of the primes in the list. [7]
- (b) By considering factorisations of the element 4 into irreducibles and using the norm $N(a + b\sqrt{-3}) = a^2 + 3b^2$, show that $\mathbb{Z}[\sqrt{-3}]$ is not a unique factorisation domain. [10]

Question 9 (17 marks, topic: metric spaces)

Define a metric on the plane, $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} d^{(2)}(\mathbf{x}, \mathbf{0}) + d^{(2)}(\mathbf{y}, \mathbf{0}), & \text{if } \mathbf{x} \neq \mathbf{y} \\ 0, & \text{if } \mathbf{x} = \mathbf{y}. \end{cases}$$

(You do not need to verify that d is a metric.)

- (a) Write down $d((1, 0), (1, t))$ and $d((0, 0), (0, t))$ for $t \neq 0$. [2]
- (b) Determine whether the sequences $((1, \frac{1}{n}))_{n \in \mathbb{N}}$ and $((0, \frac{1}{n}))_{n \in \mathbb{N}}$ are d -convergent. [3]
- (c) (i) Show that if $(\mathbf{a}_n)_{n \in \mathbb{N}}$ is a d -Cauchy sequence that is not eventually constant, then $(d^{(2)}(\mathbf{a}_n, (0, 0)))_{n \in \mathbb{N}}$ is a real null sequence and $\mathbf{a}_n \xrightarrow{d} (0, 0)$ as $n \rightarrow \infty$. [4]
(ii) Hence, or otherwise, prove that $(\mathbb{R}^2, d)$ is a complete metric space. [2]
- (d) Use the result of (c)(i) together with the fact that convergent sequences are Cauchy to find $\text{Cl}_{(\mathbb{R}^2, d)}(\{(x, y) \in \mathbb{R}^2 : y < 0\})$. [3]
- (e) Determine whether $\mathbb{R}^2 - \{(0, 0)\}$ is d -connected. [3]

[END OF QUESTION PAPER]